I. INTRODUCTION

1. Overview of the Thesis

In this thesis, I discuss the metric-connection theories of gravity. These theories generalize Einstein's theory beyond the metric theories by using some part of the connection as an independent gravitational variable in addition to the metric. Except while laying the mathematical background, I restrict my attention to Cartan connections (which are metric-compatible but have arbitrary torsion).

A typical metric-connection theory is that based on the gravitational Lagrangian,

$$L_G = \hat{R} + \hat{R}^{\alpha}{}_{\beta\gamma\delta}\,\hat{R}^{\beta}{}_{\alpha}{}^{\gamma\delta} \quad ,$$

where $\hat{R}^{\alpha}{}_{\beta\gamma\delta}$ is the curvature of a Cartan connection. This theory is studied in some detail in Chapter VI. In particular, I prove a Birkhoff theorem. Other possible gravitational Lagrangians are discussed in Chapter V.

Some experimental consequences of the metric-connection theories are examined in Chapter IV. Specifically, in a metric and torsion background spacetime, I derive propagation equations for the momentum and angular momentum of a body with both orbital angular momentum and a net elementary particle spin angular momentum. These equations show that the spin feels the torsion while the orbital angular momentum does not. For fixed background metric and torsion fields, these results are independent of the choice of gravitational Lagrangian since the propagation equations follow from the conservation laws (rederived in Section III.5) which in turn are derived via Noether's theorem from the transformation properties of the matter Lagrangian under spacetime symmetries.

In Chapter II, I argue that the metric-connection theories of gravity may be regarded as the gauge theories of the spacetime symmetries. This analogy is reinforced by the two tangent space formalism developed in Chapter III.

Before giving a more detailed outline of the thesis and summarizing the results (Section 3 of this Introduction), it is useful to discuss (Section 2 below) the reasons for investigating metric-connection theories, their historical development, and their relationship to the metric theories.

2. Metric-Connection Theories as Generalizations of the Metric Theories

Since Einstein's theory agrees with all present experiments, I wish to explain why I feel it is worthwhile to consider metric-connection theories (or any other new theories of gravity). First and most obvious, if Einstein's theory is ever found to disagree with experiment, the new theories would be ready.

Second, alternate theories may suggest new ways of testing gravity theories. In particular, in Section IV.4, I show that the metric-Cartan connection theories, unlike Einstein's, predict that the propagation of elementary particle spin angular momentum is different from the propagation of orbital angular momentum. Although the measurement of this difference is beyond present capability (as shown in Section IV.5), it may be possible at some future time when we are able to maneuver large masses (10^7 kg) in space.

Third, although Einstein's theory is amazing in its beauty and simplicity, alternate theories may provide a more aesthetic unification of the theory of gravity with the theories of the weak, electromagnetic and strong interactions. In particular, as explained in Chapter II, I feel that the metric-connection theories of gravity are in close analogy with the gauge theories of elementary particle interactions. The two tangent space formalism, presented in Chapter III, makes this analogy even closer.

Finally, alternate theories may lead to a unification of gravity with quantum mechanics. At present neither Einstein's theory nor any other theory of gravity has been quantized. This lack of a quantum theory of gravity is the most blatant deficiency in the laws of physics as understood since the days of Einstein. I consider the search for a quantum theory of gravity to be the most important reason to investigate alternate theories. With the notable exception of supergravity, to date most of the attempts at quantizing gravity have been confined to the metric theories. Since Yang-Mills fields are quantizable, the analogy between metric-connection theories and gauge theories lends hope that some metric-connection theory may prove quantizable.

To be acceptable as a physical theory, any eventual quantum theory of gravity must have a satisfactory classical limit. In fact since it may never be possible to measure the quantum mechanical properties of the gravitational field, the existence of a satisfactory classical limit should be a prerequisite to any attempt at quantization. Hence, in this thesis I concentrate on the classical properties of the metric-connection theories rather than their quantizability.

Einstein's [1915] theory of gravity is the principal example of the metric theories of gravity. In these theories the gravitational field is completely described by a metric, g, on spacetime. To be compatible with both conventions on the signature of the metric, I let $s = \pm 1$ denote the length of a unit timelike vector. From the metric one defines the metric determinant, $\tilde{g}$; the Christoffel connection, $\{^a{}_{bc}\}$; the Christoffel Riemann curvature, $\tilde{R}^a{}_{bcd}$; the Christoffel Ricci curvature, $\tilde{R}_{bd}$; the Christoffel scalar curvature, $\tilde{R}$; and the Christoffel Einstein curvature, $\tilde{G}_{bd}$. (See the Appendix on Notation for definitions.)

Before discussing other metric theories, I first review some properties of Einstein's theory. Its field equations are

$$\tilde{G}^{ab} = \frac{8\pi L^2}{\hbar c} \tilde{T}^{ab} , \tag{1}$$

where $L = (G\hbar/c^3)^{\frac{1}{2}}$ is the Planck length and $\tilde{T}^{ab}$ is the metric energy-momentum tensor which may be defined either phenomenologically or in terms of a matter Lagrangian, L_M, according to the formula

$$- s \sqrt{-\tilde{g}} \, \frac{1}{2} \tilde{T}^{ab} = \frac{\delta \mathcal{L}_M}{\delta g_{ab}} , \tag{2}$$

where $\mathcal{L}_M = \sqrt{-\tilde{g}} \, L_M$. If there is a matter Lagrangian which is a scalar and if the matter field equations are satisfied, then Noether's theorem implies that $\tilde{T}^{ab}$ satisfies the energy-momentum conservation laws:

$$\nabla_b \tilde{T}^{ab} = 0. \tag{3}$$

However, even if there is no matter Lagrangian so that $\tilde{T}^{ab}$ must be defined phenomenologically, the conservation laws (3) are still satisfied by virtue of the field equations (1) and the identity,

$$\nabla_b \tilde{G}^{ab} = 0 , \tag{4}$$

satisfied by the Christoffel Einstein tensor. Consequently, one says that Einstein's theory has automatic Noether conservation laws. In fact the existence of automatic conservation laws was one of the primary reasons why Einstein chose his field equations in the first place.

Many authors, beginning with Einstein [1917] and Eddington [1924], have modified Einstein's theory within the context of metric theories while retaining its automatic conservation laws. Thus they replace Einstein's equations (1) by equations of the form

$$\tilde{E}^{ab} = \tilde{T}^{ab} \quad , \tag{5}$$

where $\tilde{E}^{ab}$ is some function of the metric and its derivatives to some order and where $\tilde{E}^{ab}$ is required to satisfy an identity of the form

$$\nabla_b \tilde{E}^{ab} = 0. \tag{6}$$

The field equations (5), together with the identity (6), automatically imply the conservation laws (3).

For example, Einstein's [1917] theory with a cosmological constant, Λ, has the field equations

$$\frac{\hbar c}{8\pi L^2} (\tilde{G}^{ab} - s \quad \Lambda\, g^{ab}) = \tilde{T}^{ab}. \tag{7}$$

Since the Christoffel connection is metric-compatible, the metric satisfies

$$\nabla_b g^{ab} = 0. \tag{8}$$

This identity and the identity (4), together with the field equations (7) imply the conservation laws (3).

Tensors $\tilde{E}^{ab}$ satisfying (6) are easy to construct. For any scalar, $L_G(g_{ab}, \partial_c g_{ab}, \partial_d \partial_c g_{ab})$, (the gravitational Lagrangian) define $\tilde{E}^{ab}$ by the formula

$$s \sqrt{-\tilde{g}}\, \frac{1}{2}\, \tilde{E}^{ab} = \frac{\delta \mathcal{L}_G}{\delta g_{ab}} = \frac{\partial \mathcal{L}_G}{\partial g_{ab}} - \partial_c \frac{\partial \mathcal{L}_G}{\partial \partial_c g_{ab}} + \partial_d\, \partial_c \frac{\partial \mathcal{L}_G}{\partial \partial_d \partial_c g_{ab}}\,, \tag{9}$$

where $\mathcal{L}_G = \sqrt{-\tilde{g}}\, L_G$. A proof similar to that for Noether's theorem shows that this $\tilde{E}^{ab}$ satisfies the identity (6). Hence, the theory with the field equations (5) with $\tilde{E}^{ab}$ defined by (9), has automatic Noether conservation laws.

Examining the definitions (2) and (9) one sees that if there is both a matter Lagrangian, L_M, and a gravitational Lagrangian, L_G, then the field equations (5) can be derived from the variational principle

$$\delta \int (L_G + L_M) \sqrt{-\tilde{g}}\, d^4x = 0. \tag{10}$$

In particular, the field equations (1) of Einstein's theory may be derived using Hilbert's [1915] gravitational Lagrangian:

$$L_G = - s \frac{\hbar c}{16\pi L^2} \tilde{R}. \tag{11}$$

The field equations (7) of Einstein's theory with a cosmological constant follow from the gravitational Lagrangian:

$$L_G = - \frac{\hbar c}{8\pi L^2} \Lambda - s \frac{\hbar c}{16\pi L^2} \tilde{R}. \tag{12}$$

This is the most general scalar which is a linear polynomial in the Christoffel curvature.

Eddington [1924] considered the Lagrangians

$$L_G = \tilde{R}_{ab}\,\tilde{R}^{ab}\,, \tag{13}$$

$$L_G = \tilde{R}_{abcd}\,\tilde{R}^{abcd}\,. \tag{14}$$

Lanczos [1938] (and Weyl [1919], but in a non-metric theory,) considered these Lagrangians and also

$$L_G = \tilde{R}\,\tilde{R}\,. \tag{15}$$

Any scalar which is quadratic in the Christoffel curvature must be a linear combination of Lagrangians (13), (14) and (15). However, Lanczos showed that in constructing field equations only two of the three scalars are required. This follows because the Euler-Gauss-Bonnet integral,

$$\int \varepsilon_{abkl}\,\varepsilon^{cdmn}\,\tilde{R}^{ab}{}_{cd}\,\tilde{R}^{kl}{}_{mn}\,\sqrt{-\tilde{g}}\,d^4x$$

$$= -4\int (\tilde{R}\,\tilde{R} - 4\,\tilde{R}_{ab}\,\tilde{R}^{ab} + \tilde{R}_{abcd}\,\tilde{R}^{abcd})\,\sqrt{-\tilde{g}}\,d^4x, \tag{16}$$

is a topological invariant. Most authors have chosen to work with the scalars $\tilde{R}_{ab}\,\tilde{R}^{ab}$ and $\tilde{R}\,\tilde{R}$ because they involve only the Ricci curvature and so are easier to work with. Unfortunately, the field equations for these quadratic curvature Lagrangians involve higher than second derivatives of the metric.

Havas [1977] has shown that any field equations (5) in which $\tilde{E}^{ab}$ involves higher than second derivatives of the metric leads to a theory which either has a bad Newtonian limit or has extraneous spherically symmetric solutions. At the same time Lovelock [1972] has demonstrated that any tensor, $\tilde{E}^{ab}$, involving no higher than second derivatives of the metric and satisfying (6), must be a linear combination of $\tilde{G}^{ab}$ and g^{ab}. Thus it appears that Einstein's theory with a cosmological constant which has the field equations (7) is the only acceptable metric theory of gravity.

On the other hand, there are reasons for including quadratic curvature terms in the Lagrangian or including higher derivatives of the metric in the field equations based on recent work on the problem of unifying gravity with quantum mechanics. First, in renormalizing the energy-momentum tensor, $\tilde{T}^{cb}$, of a quantum matter field in a curved classical background spacetime, it is found that there are counterterms added to the Einstein equations (1) which involve third and fourth derivatives of the metric. Second, while Einstein's theory with the Lagrangian (11) is unitary but non-renormalizable, Stelle's [1977] theory with the Lagrangian,

$$L_G = - s \frac{\hbar c}{16\pi L^2} \tilde{R} - \alpha \tilde{R}_{ab} \tilde{R}^{ab} + \beta \tilde{R} \tilde{R}, \qquad (17)$$

is renormalizable but non-unitary. One hopes to eventually find a theory which is both unitary and renormalizable. And third, the scalar $\tilde{R}^a{}_{bcd} \tilde{R}^b{}_a{}^{cd}$ is analogous to the Yang-Mills Lagrangian $F^P{}_{cd} F_P{}^{cd}$. Since the Yang-Mills theory is quantizable, there is hope that the use of this scalar as the gravitational Lagrangian or as a term in the gravitational Lagrangian will help in the quantization of gravity.

Can we find a way to include quadratic curvature terms in the Lagrangian which maintains the good classical behavior of Einstein's theory?

There is one way in which I feel the above criteria for choosing the gravitational field equations ought to be weakened. In the presence of particles with spin, the definition of the energy-momentum tensor becomes ambiguous and the conservation law (3) may no longer be appropriate. Instead of (3) one might use both the alternate energy-momentum conservation law,

$$\nabla_a t_c{}^a = \frac{1}{2} S^b{}_a{}^d \tilde{R}^a{}_{bcd} \ , \tag{18}$$

and the angular momentum conservation law,

$$\nabla_c S_{ba}{}^c = t_{ba} - t_{ab} \ , \tag{19}$$

where $t_b{}^a$ is the canonical energy-momentum tensor (asymmetric) and $S^b{}_a{}^c$ is the canonical spin tensor (antisymmetric in b and a). The tensors $t_b{}^a$ and $S^b{}_a{}^c$ may be defined either phenomenologically or in terms of a matter Lagrangian, L_M, according to the formulas

$$- s \sqrt{-\tilde{g}}\, t_\alpha{}^a = \left.\frac{\delta \mathcal{L}_M}{\delta \theta^\alpha{}_a}\right|_{\{^\alpha{}_{\beta c}\}} \ , \tag{20}$$

$$- s \sqrt{-\tilde{g}}\, \frac{1}{2} S^\beta{}_\alpha{}^c = \left.\frac{\delta \mathcal{L}_M}{\delta \{^\alpha{}_{\beta c}\}}\right|_{\theta^\alpha{}_a} \ , \tag{21}$$

where $\mathcal{L}_M = \sqrt{-\tilde{g}}\ L_M$ is regarded as an explicit function of the components of an orthonormal 1-form frame, $\theta^\alpha = \theta^\alpha{}_a\, dx^a$, and the mixed components of the Christoffel connection, $\{^\alpha{}_{\beta c}\} = \theta^\gamma{}_c \{^\alpha{}_{\beta\gamma}\}$. (See the Appendix on Notation.) The conservation laws (18) and (19) are only appropriate to the metric theories. When there is a matter Lagrangian, they may be derived via Noether's theorem. (The metric-connection theory generalizations of (18) and (19) are derived in Section III.5.)

The metric energy-momentum tensor, $\tilde{T}^{ab}$, may be expressed in terms of $t_b{}^a$ and $S^b{}_a{}^c$ according to the Belinfante [1940] and Rosenfeld [1940] symmetrization procedure:

$$\begin{aligned}\tilde{T}^{ab} &= t^{ab} - \frac{1}{2} \nabla_c (S^{abc} + S^{cab} + S^{cba}) \\ &= t^{(ab)} - \nabla_c S^{c(ab)} \ .\end{aligned} \tag{22}$$

The conservation law (3) is still true in the presence of spin but it does not contain as much information as (18) and (19).

From the above discussion of the conservation laws satisfied by $t_b{}^a$, $S^b{}_a{}^c$ and $\tilde{T}^{ab}$, I wish to draw exactly the opposite conclusion as that drawn by Belinfante. He says, "However, it cannot be expected that $t^{\mu\nu}$ will represent the 'true' energy and momentum current and density. ... we should regard, in the present case, only *that* current-density tensor $T^{\mu\nu}$, that generates the *gravitational* field, as the *true* energy tensor." (Italics are Belinfante's.) Since $\tilde{T}^{ab}$ is the source in Einstein's equations, Belinfante concludes that $\tilde{T}^{ab}$ is the true energy-momentum tensor. However, since $t_b{}^a$ contains more information than $\tilde{T}^{ab}$, I wish to regard $t_b{}^a$ as the true energy-momentum tensor. Hence, I conclude that *one ought to modify Einstein's equations so that* $t_b{}^a$ *is the source.*

Thus one should look for a new set of gravitational field equations of the form

$$E_b{}^a = t_b{}^a \ , \tag{23}$$

where $E_b{}^a$ is a function of the metric and its derivatives to some order. Since $t_b{}^a$ should satisfy the conservation laws (18) and (19), it would be wrong to assume that $E_b{}^a$ is either divergence-free or symmetric. The former and (18) would impose a severe restriction on the curvature and spin. The latter and (19) would imply separate conservation of spin and orbital angular momentum.

The trouble with equation (23) is that there is now no way to derive the conservation laws (18) and (19) for phenomenologically defined $t_b{}^a$ and $S^b{}_a{}^c$. Perhaps one should introduce an additional set of gravitational field equations of the form

$$C^b{}_a{}^c = S^b{}_a{}^c \ , \tag{24}$$

where $C^{b\ c}_{\ a}$ is also a function of the metric and its derivatives to some order. Then by requiring $E_b^{\ a}$ and $C^{b\ c}_{\ a}$ to satisfy identities of the form

$$\nabla_a E_c^{\ a} = \frac{1}{2} C^{b\ d}_{\ a} \tilde{R}^a_{\ bcd} , \tag{25}$$

$$\nabla_c C_{ba}^{\ \ c} = E_{ba} - E_{ab} , \tag{26}$$

the field equations (23) and (24) would automatically imply the conservation laws (18) and (19).

By analogy with equation (22), one could define

$$\tilde{E}^{ab} = E^{(ab)} - \nabla_c C^{c(ab)} . \tag{27}$$

Then equations (22), (27), (23) and (24) imply the equation,

$$\tilde{E}^{ab} = \tilde{T}^{ab} , \tag{28}$$

while the identities (25) and (26) imply the identity;

$$\nabla_b \tilde{E}^{ab} = 0. \tag{29}$$

Equation (28) with identity (29) coincides with the field equations (5) with identity (6). However, (28) does not contain as much information as both of equations (23) and (24).

As before, it is easy to construct tensors $E_b^{\ a}$ and $C^{b\ c}_{\ a}$ satisfying (25) and (26). For any scalar gravitational Lagrangian,

$$L_G(\theta^\alpha_{\ a}, \{^\alpha_{\ \beta c}\}, \partial_d\{^\alpha_{\ \beta c}\}),$$

define $E_b^{\ a}$ and $C^{b\ c}_{\ a}$ by the formulas

$$s\,\sqrt{-\tilde{g}}\;\; E_\alpha{}^a = \left.\frac{\delta\mathcal{L}_G}{\delta\theta^\alpha{}_a}\right|_{\{^\alpha{}_{\beta a}\}} = \frac{\partial\mathcal{L}_G}{\partial\theta^\alpha{}_a}\,, \tag{30}$$

$$s\,\sqrt{-\tilde{g}}\;\;\frac{1}{2}\,C^\beta{}_\alpha{}^c = \left.\frac{\delta\mathcal{L}_G}{\delta\{^\alpha{}_{\beta c}\}}\right|_{\theta^\alpha{}_a} = \frac{\partial\mathcal{L}_G}{\partial\{^\alpha{}_{\beta c}\}} - \partial_d\,\frac{\partial\mathcal{L}_G}{\partial\partial_d\{^\alpha{}_{\beta c}\}}\,, \tag{31}$$

where $\mathcal{L}_G = \sqrt{-\tilde{g}}\, L_G$. A proof similar to that for Noether's theorem again shows that this $E_b{}^a$ and $C^b{}_a{}^c$ satisfy the identities (25) and (26). Hence, the theory with the field equations (23) and (24) with $E_b{}^a$ and $C^b{}_a{}^c$ defined by (30) and (31) has automatic Noether conservation laws (18) and (19).

Unfortunately, there is still a problem with equations (23) and (24). In order to attempt a path-integral quantization of gravity, it is necessary to have an action functional from which the field equations can be derived by a variational principal. Equations (23) and (24) cannot be derived from a variational principle (at least not straightforwardly). To see this, suppose there is both a matter Lagrangian, L_M, and a gravitational Lagrangian, L_G. An examination of definition (20), (21), (30) and (31) naively appears to say that the field equations (23) and (24) can be derived from the variational principle,

$$\delta \int \;(L_G + L_M)\,\sqrt{-\tilde{g}}\; d^4x = 0, \tag{32}$$

by varying $\theta^\alpha{}_a$ and $\{^\alpha{}_{\beta c}\}$ independently. This is essentially a "Palatini"[1919] variation. The problem is that $\theta^\alpha{}_a$ and $\{^\alpha{}_{\beta c}\}$ cannot be varied independently since $\{^\alpha{}_{\beta c}\}$ can be expressed in terms of $\theta^\alpha{}_a$ and its derivatives according to the formulas

$$\{^{\alpha}{}_{\beta c}\} = \frac{1}{2}\,\theta^{\gamma}{}_{c}\; g^{\alpha\delta}(c_{\beta\delta\gamma} + c_{\gamma\delta\beta} - c_{\delta\beta\gamma}), \tag{33}$$

$$c^{\alpha}{}_{\beta\gamma} = e_{\beta}{}^{b}\; e_{\gamma}{}^{c}(\partial_{c}\;\theta^{\alpha}{}_{b} - \partial_{b}\;\theta^{\alpha}{}_{c}). \tag{34}$$

When this interdependence is taken into account, the variational principle (32) leads to only the field equation (28) which is implied by equations (23) and (24) but does not contain as much information.

Thus equations (23) and (24) can be used as a priori field equations, but they cannot easily be derived from an action principle. Can one find an action for which the field equations have both $t_b{}^a$ and $S^b{}_a{}^c$ as sources and have automatic conservation laws? Yes, if one drops the restriction to metric theories and makes the connection independent.

By a metric-connection theory of gravity, I mean a theory in which the gravitational field is completely described by a metric and a connection which may be non-metric-compatible and may have torsion. Thus the covariant derivative of the metric

$$\nabla_{\gamma} g_{\alpha\beta} = e_{\gamma} g_{\alpha\beta} - \Gamma^{\delta}{}_{\alpha\gamma}\; g_{\delta\beta} - \Gamma^{\delta}{}_{\beta\gamma}\; g_{\alpha\delta} \;, \tag{35}$$

and/or the torsion

$$Q^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta} - \Gamma^{\alpha}{}_{\beta\gamma} - c^{\alpha}{}_{\beta\gamma} \;, \tag{36}$$

may be non-zero. Although all metric theories (which use the Christoffel connection) are metric-connection theories, I am mainly concerned with those theories in which the connection cannot be completely specified in terms of the metric. The principal examples of the metric-connection theories are Cartan's [1922, 23, 24, 25] theory and Weyl's [1919, 21] theory.

3. Outline of the thesis

In the first third of the thesis (Chapters II and III), I concentrate on the kinematics of a metric-connection gravitational field. I first discuss the analogy between the gauge theories of elementary particle interactions (Section II.1) and the metric-connection theories of gravity (Section II.2). Then I discuss the possible space-time symmetry groups (Section II.3) and correlate each of these with a class of "admissible" frames (Section II.3) and with a set of restrictions on the connection (Section II.4). See Tables II.1 through II.5. For example, for the groups O(3,1,R) and SL(2,C), the admissible tangent frames are orthonormal, the admissible spinor frames are orthonormal, and the connection is a Cartan connection. Similarly, the group, GL(4,R) admits arbitrary tangent frames and allows a general connection but is incompatible with spinors. On the other hand, the group, GL(2,C), admits arbitrary spinor frames, requires conformal orthonormal tangent frames, and unifies the electromagnetic potential with a Weyl-Cartan connection.

In Section II.4, I also define the mixed components of the connection, $\Gamma^{\alpha}{}_{\beta a}$, and use the gauge theory analogy to justify choosing the components of the admissible 1-form frame, $\theta^{\alpha} = \theta^{\alpha}{}_{a}\, dx^{a}$, the frame components of the metric, $g_{\alpha\beta}$, and the mixed components of the connection, $\Gamma^{\alpha}{}_{\beta a}$, as the best variables for describing the gravitational field.

In Chapter III, I introduce mixed covariant derivatives such as

$$\nabla_a t_{\hat{\alpha}}{}^{\tilde{b}} = \partial_a t_{\alpha}{}^{b} - \Gamma^{\beta}{}_{\alpha a}\, t_{\beta}{}^{b} + \{^{b}{}_{ca}\}\, t_{\alpha}{}^{c} \quad ,$$

where some indices (those with a caret, ^) are corrected with the full connection, while other indices (those with a tilde, ~) are corrected with the Christoffel connection. These mixed covariant derivatives appear in many of the equations of the metric- connection theories, and it is not immediately obvious why some indices are corrected with full connections and other indices are corrected with Christoffel connections. I develop a two tangent space formalism (Section III.2) to give a geometric explanation for the presence of two connections and two types of indices. The Christoffel connection acts on one of the tangent spaces (called the external tangent space) where I usually use coordinate indices. The full connection acts on the other tangent space (called the internal tangent space) where I usually use orthonormal indices.

As discussed in Section III.4, all differentiating directions belong to the external tangent space. This implies that the differentiating indices on all connections, curvatures, gauge potentials, and gauge fields are external indices. All other tangent indices are standardly taken as internal tangent indices. In particular, all matter fields except gauge fields are standardly internal tangent tensors.

There is an isomorphism (called the soldering isomorphism) between the two tangent spaces (Section III.2). Its covariant derivative turns out to be the defect tensor which is the difference between the full connection and the Christoffel connection. The soldering isomorphism and its covariant derivative turn out to be useful tools in computation.

In Section III.3, I compare the two tangent space formalism with the usual one tangent space formalism. In particular, the soldering isomorphism and the external orthonormal frame both correspond to the orthonormal frame when there is only one tangent space. I easily generalize the Cartan differential form notation from one to two tangent spaces. The form indices belong to the external tangent form bundles.

Around the middle third of the thesis (Chapter IV and parts of Chapters II and III), I concentrate on the kinematics and dynamics of matter fields in a metric-connection spacetime. In Section II.3, I review the transformation properties of various fields under coordinate, tangent frame, and spinor frame transformations (Tables II.6 through II.11), and in Section III.4, I discuss minimal coupling of the matter fields to the gravitational field.

As an example of a computation using the two tangent space formalism, I rederive (Section III.5) the conservation laws of energy-momentum and angular momentum by applying Noether's theorem to coordinate and O(3,1,R)-frame invariance. Considering GL(4,R)-frame invariance, I also obtain conservation laws for hypermomentum and dilation current.

From the conservation laws of energy-momentum and angular momentum, William Stoeger and I have derived (Sections IV.2 and IV.3) propagation equations for the integrated energy-momentum and integrated angular momentum of a body with both orbital angular momentum and elementary particle spin angular momentum, moving in a metric-Cartan connection spacetime. These propagation equations show (Section IV.4) that the torsion couples to the spin angular momentum but not to the orbital angular momentum. However, experiments (Section IV.5) to measure the effect of the torsion on the spin are beyond present technology.

The method of deriving the propagation equations is similar to that used by Papapetrou [1951] and Dixon [1970a,b; 1974] in the context of metric theories. However, they did not regard the spin and orbital angular momentum as separate quantities. As a special case of our propagation equations, we show (Section IV.4, Corollary IV.7) that in a metric theory, spin and orbital angular momentum propagate in the same way.

To summarize my treatment of the behavior of matter in metric-Cartan connection spacetimes, there is a progression from the transformation properties of fields under spacetime symmetries (Section II.3 and III.4), to the differential conservation laws of energy-momentum and angular momentum (Section III.5), to the propagation equations for the integrated energy-momentum and angular momentum (Sections IV.1 through IV.4), and to conceivable experiments to detect the torsion (Section IV.5).

In the final third of the thesis (Chapters V and VI) I discuss the dynamics of the gravitational field, i.e. the choice of gravitational Lagrangian or gravitational field equations. A viable theory of gravity must agree with Newtonian experiments, post-Newtonian solar system experiments and cosmological observations (Section V.2). It should also have a good initial value formulation, have automatic Noether conservation laws and ultimately be quantizable (Section V.3).

Skinner and Gregorash [1976] and Aldersley [1977 a,b] have studied the class of metric-Cartan connection theories in which the field equations contain no higher than second derivatives of the metric and torsion. I obtain (Section V.3b) a different class of metric-Cartan connection theories by requiring the field equations to contain no higher than second derivatives of the components of the orthonormal 1-form frame, $\theta^{\alpha}{}_{a}$, and the mixed components of the Cartan connection, $\Gamma^{\alpha}{}_{\beta a}$. This class contains all theories derived from a gravitational Lagrangian of the form

$$L_G = \alpha\, \tilde{R} + \bar{L}\, (g_{\alpha\beta},\, \hat{R}^{\alpha}{}_{\beta\gamma\delta},\, Q^{\alpha}{}_{\gamma\delta}),$$

where $\tilde{R}$ is the scalar curvature of the Christoffel connection, α is an arbitrary constant, and $\bar{L}$ is an arbitrary scalar function of the metric, $g_{\alpha\beta}$, the Cartan curvature, $\hat{R}^{\alpha}{}_{\beta\gamma\delta}$, and the torsion, $Q^{\alpha}{}_{\gamma\delta}$.

This type of Lagrangian describes a large class of theories. I show (Section V.3c) that all of them have automatic Noether conservation laws. Then I show (Section V.3d) that any such Lagrangian which is a quadratic polynomial in $\hat{R}^{\alpha}{}_{\beta\gamma\delta}$ and $Q^{\alpha}{}_{\gamma\delta}$, must belong to the twelve parameter family

$$L_G = -\frac{\hbar c}{8\pi L^2}\,\Lambda - s\,\frac{\hbar c}{16\pi L^2}\,(c_1\,\hat{R} + c_2\,\tilde{R})$$

$$+\frac{\hbar c}{16\pi}\,(a_1\,\hat{R}\,\hat{R} + a_2\,\hat{R}_{\beta\delta}\,\hat{R}^{\beta\delta} + a_3\,\hat{R}_{\alpha\beta\gamma\delta}\,\hat{R}^{\alpha\beta\gamma\delta}$$

$$+ a_4\,\hat{R}_{\beta\delta}\,\hat{R}^{\delta\beta} + a_5\,\hat{R}_{\alpha\beta\gamma\delta}\,\hat{R}^{\gamma\delta\alpha\beta} + a_6\,\hat{R}_{\alpha\beta\gamma\delta}\,\hat{R}^{\alpha\gamma\beta\delta})$$

$$- s\,\frac{\hbar c}{16\pi L^2}\,(b_1\,Q^{\alpha}{}_{\alpha\delta}\,Q^{\gamma}{}_{\gamma}{}^{\delta} + b_2\,Q^{\alpha}{}_{\gamma\delta}\,Q_{\alpha}{}^{\gamma\delta} + b_3\,Q^{\alpha}{}_{\gamma\delta}\,Q^{\gamma}{}_{\alpha}{}^{\delta}).$$

Finally (Chapter VI) I restrict my attention to the gravitational Lagrangian

$$L_G = -s\,\frac{\hbar c}{16\pi L^2}\,\hat{R} + \frac{\hbar c}{16\pi\alpha_G}\,\hat{R}_{\alpha\beta\gamma\delta}\,\hat{R}^{\alpha\beta\gamma\delta}\,,$$

where $\hat{R}^{\alpha}{}_{\beta\gamma\delta}$ is the Cartan curvature. In Section VI.2, I rederive the automatic Noether conservation laws and point out that for this theory the frame, $\theta^{\alpha}{}_{a}$, acts as a Lagrange multiplier. Sriram Ramaswamy and I have proven (Section VI.3) a Birkhoff theorem which says that the unique O(3)-spherically symmetric vacuum solution of this theory is the Schwarzschild metric and zero torsion.